Problem 2.4

(a) In how many different ways can the letters of the word "MISSISSIPPI" be arranged if the four S's cannot appear consecutively?

Solution Note that the distinct letters are 1 M, 2 Ps, 4 Is and 4 Ss making for a total of 11. There are 11! ways to order 11 objects and accounting for the repeated letters we get

$$\frac{11!}{2!4!4!} = 34650$$

as our answer.

(b) An n-term sequence $(x_1, x_2, ..., x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let an be the number of binary sequences of length n containing no three consecutive terms equal to (010) in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to (0011) or (1100) in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Solution (Mario Tutuncu-Macias) We construct a 2-to-1 correspondence between the sequences counted by b_{n+1} and a_n. For a binary sequence $(x_1, x_2, \ldots, x_{n+1})$ form the sequence of pairwise differences

$$y = (x_2 - x_1, x_3 - x_2, \ldots x_{n+1} - x_n)$$

modulo 2. Now y is a binary sequence of length n with the following crucial property: y contains the subsequence (010) if and only if x contains at least one of (0011) and (1100). Furthermore, x uniquely defines y while given a sequence y the corresponding x can be recovered uniquely given the value of x_1. Thus, we have constructed the desired 2-to-1 correspondence thereby proving that

$$b_{n+1} = 2a_n$$

for all $n \geq 1$ as required.

Problem 2.5

In Piotrland there are no kings or queens, but there are a different tribes. A council sits on a circular table that has p seats, where p is a prime number. Councils can be formed by people from different tribes, but councils cannot be all composed of any one single tribe. How many possible councils are there?

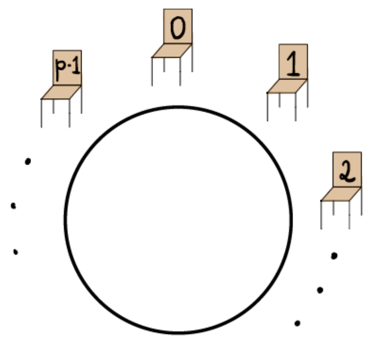

Figure 1: Seats around the table.

Solution Let us number the p seats around the table from 0 to $p-1$ as shown on Figure 1. We use tuples $x = (x_0, x_1, \ldots x_{p-1})$ to represent seatings where x_i = tribe of person sitting at seat i. Note that two seatings $x \neq y$ can correspond to the same council! We call a seating valid if at least two tribes are present. For $n \geq 0$ let $R(x, n)$ be the tuple we get after n clockwise rotations of x, i.e. $R(x, 1) = (x_1, x_2, \ldots x_{p-1}, x_0)$ and $R(x, 2) = (x_2, x_3, \ldots x_{p-1}, x_0, x_1)$ et cetera (see Figure 2).

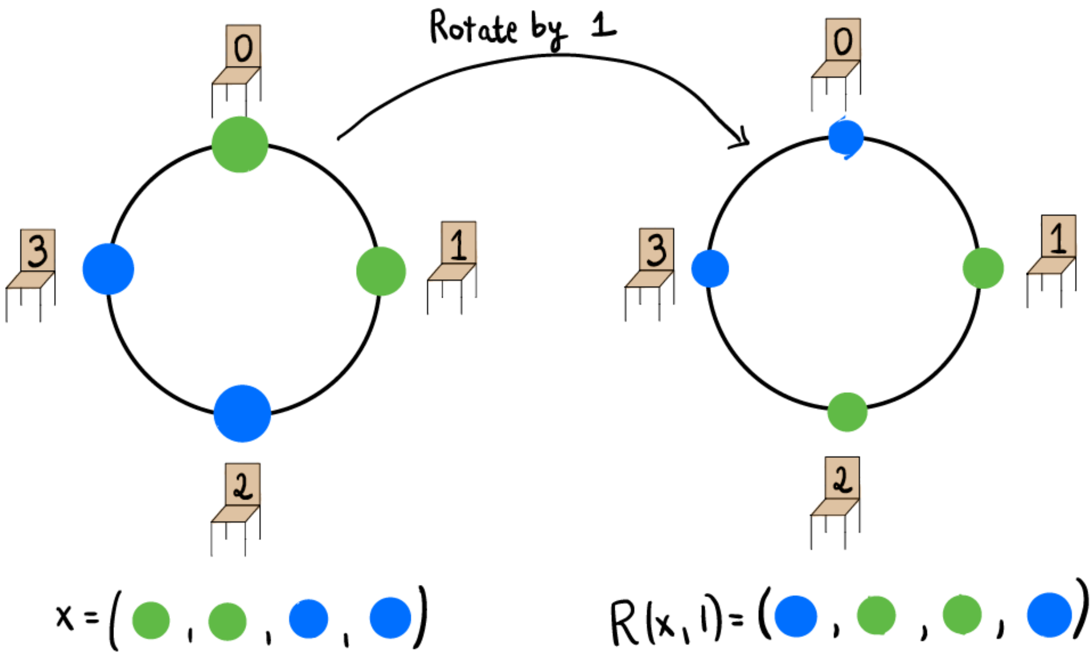

Figure 2: Rotating a seating by 1.

Finally, for a seating x let S_x be the *orbit* of x defined as

$$S_x := \{R(x,n) \,:\, n \geq 0\}.$$

In other words, S_x is the set of seatings one can obtain by rotating x. See Figure 3 for an example.

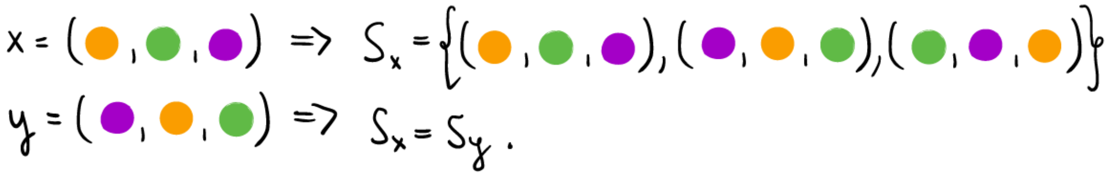

Figure 3: Example of an orbit.

Lemma 1 — *For a valid seating x and $n \geq 0$ we have $R(x,n) = x$ if and only if $n \equiv 0 \pmod{p}$.*

Remark 1. Notice that rotations by $n \geq 0$ where $n \equiv 0 \pmod{p}$ are precisely the full rotations of the table: if $n = 0$ then you don't rotate, if $n = p$ then you fully rotate once, if $n = k \cdot p$ you fully rotate k times. Thus, $R(x, kp) = x$ is not saying much. However, the lemma says that $R(x,n) = x$ can happen *only if* $n = k \cdot p$ for some k.

Proof. Suppose that $R(x,n) = x$. Looking at the first entry of the two tuples this gives

$$x_{n \pmod{p}} = x_0.$$

Note that clearly $x = R(x,n) = R(R(x,n),n) = R(x,2n)$. Continuing like this, by induction, $x = R(x, k \cdot n)$ for all $k \geq 0$. We have shown that

$$x_0 = x_{n \pmod{p}} = x_{2n \pmod{p}} = x_{3n \pmod{p}} = \dots. \tag{1}$$

If $n \equiv 0 \pmod{p}$ this just says that $x_0 = x_0 = x_0 = \dots$. If $n \not\equiv 0 \pmod{p}$ however, then n and p are relatively prime, and thus

$$\{0, n \pmod{p}, 2n \pmod{p}, (p-1)n \pmod{p}\} = \{0, 1, 2, \dots p-1\}.$$

For example, for $p = 5$ and $n = 7$

$$\{0, 7 \,(\mathrm{mod}\ 5), 2 \cdot 7 \,(\mathrm{mod}\ 5), 3 \cdot 7 \,(\mathrm{mod}\ 5), 4 \cdot 7 \,(\mathrm{mod}\ 5)\} = \{0, 2, 4, 1, 3\} = \{0, 1, 2, 3, 4\}.$$

We see now from equation (1) that $n \not\equiv 0 \pmod{p}$ would contradict the fact that x is a valid seating. Therefore the only possibility is to have $n \equiv 0 \pmod{p}$. ☐

What Lemma 1 tells us is that

$$|S_x| = p \tag{2}$$

for all valid seatings x. Indeed, by its definition the orbit of x is

$$S_x = \{x, R(x,1), R(x,2), \dots R(x, p-1)\}$$

and the elements on the RHS above are all distinct by Lemma 1. Don't be fooled by the simplicity of the fact (2). It only works because p was prime! See Figure 4 for a counterexample when $p = 4$.

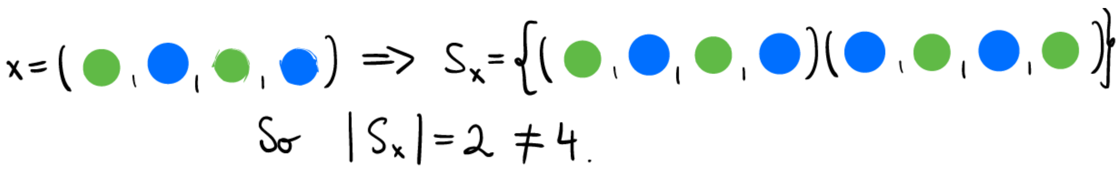

Figure 4: Example of an orbit with $|S_x| \neq p$.

Recall that our goal is to count the number of councils that can be formed. The crucial observation is that

$$\text{number of councils } = \text{number of distinct orbits} = |\{S_x \,:\, x \text{ is a valid seating}\}|.$$

For two valid seatings x, y it is easy to see that either $S_x = S_y$ or $S_x \cap S_y = \varnothing$. In other words, the distinct orbits form a partition of the set of valid seatings. Take valid seatings $x^{(1)}, x^{(2)}, \ldots x^{(l)}$ such that their orbits $S_{x^{(1)}}, \ldots S_{x^{(l)}}$ are distinct and

$$\bigcup_{i=1}^{l} S_{x^{(i)}} = \{\text{all valid seatings}\}.$$

Note that necessarily $l =$ number of distincts orbits $=$ number of councils. Taking the cardinality of both sets, we obtain

$$\left|\bigcup_{i=1}^{l} S_{x^{(i)}}\right| = \text{number of valid seatings.} \tag{3}$$

We know that for disjoint sets A, B we have $|A \cup B| = |A| + |B|$ so that the LHS above is

$$\left|\bigcup_{i=1}^{l} S_{x^{(i)}}\right| = \sum_{i=1}^{l} |S_{x^{(i)}}| = l \cdot p.$$

What remains is counting the number of valid seatings. But this is easy! There are p seats and a tribes, so in total there are a^p seatings. However, a of these are invalid because only one tribe is present. Therefore, in total there are $a^p - a$ valid seatings. Putting everything together, plugging into equation (3) we get

$$l \cdot p = a^p - a,$$

so that our final answer is

$$\text{number of councils} = \frac{a^p - a}{p}.$$

Concluding remarks In the process of solving the problem we actually proved a well known theorem in number theory, namely Fermat's Little Theorem. More generally, you may wonder what happens to our solution if p is not prime. This more difficult case can be tackled with a slight generalization of our argument above which is known as Burnside's Lemma, an awesome theorem from the field of Group Theory.